# ALGEBRAIC CURVES
# SOLUTION SHEET 3

Unless otherwise specified, $k$ is an algebraically closed field.

**Exercise** 3.1.

(1) Show that $V(Y - X^2) \subset \mathbb{A}^2(\mathbb{C})$ is irreducible; in fact, $I(V(Y - X^2)) = (Y - X^2)$.
(2) Decompose $V(Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3) \subset \mathbb{A}^2(\mathbb{C})$ into irreducible components.
(3) Show that $F = Y^2 + X^2(X-1)^2 \in \mathbb{R}[X, Y]$ is an irreducible polynomial, but $V(F)$ is reducible.

**Solution 1.**

(1) If we can show that $I \coloneqq (Y - x^2)$ is prime then both statements follow. Now $I = (Y - X^2)$ is prime because $\mathbb{C}[X, Y]/I \simeq \mathbb{C}[X]$ is integral (to see the isomorphism, consider the map $\mathbb{C}[X, Y] \to \mathbb{C}[X]$ sending $X$ to $X$ and $Y$ to $X^2$).
(2) Let $V := V(Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3)$. We can see that

$$Y^4 - X^2 = (Y^2 + X)(Y^2 - X)$$

and

$$Y^4 - X^2Y^2 + XY^2 - X^3 = (Y^2 + X)(Y^2 - X^2)$$

We see that $Y^2 + X$ is a common irreducible factor so it is an irreducible component of dimension 1 in $V$. Indeed, if $I \coloneqq (Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3)$, $I_1 \coloneqq (Y^2 + X)$ and $I_2 \coloneqq (Y^2 - X, Y^2 - X^2)$, we have $I = I_1 \cdot I_2$ (the RHS is interpreted as the ideal generated by $\{i_1 i_2 \mid i_1 \in I_1,\ i_2 \in I_2\}$), so by Exercise 2.3.2 we have $V(I) = V(I_1) \cup V(I_2)$. By a similar argument as in part 1, we have that $V(I_1)$ is irreducible. As $V(I_1) \subsetneq V(I_2)$ (we have e.g. $(i, 1) \in V(I_1) \backslash V(I_2)$), it follows that $V(I_1)$ is an irreducible component of $V(I)$, and as $I_1$ is a height 1 prime ideal in $\mathbb{C}[X, Y]$, it follows that $V(I_1)$ has codimension 1 in $\mathbb{A}^2(\mathbb{C})$, i.e. it has dimension 1. There are two other irreducible components given by points $(1, 1)$ and $(1, -1)$ in the intersection of $V(Y^2 - X)$ and $V(Y^2 - X^2) = V((Y - X)(Y + X))$. (Note that $(0, 0)$ is already contained in $V(Y^2 + X)$)
(3) We view $F$ as an element of $\mathbb{R}[X][Y]$. As $F$ has degree 2 in $Y$, it is reducible if and only if there exists $p \in \mathbb{R}[X]$ such that $F(X, p(X)) = 0$.

This is impossible as $F(-1, p(-1)) > 0$. Alternatively, in $\mathbb{C}[X, Y]$,

$$F = (Y - iX(X-1))(Y + iX(X-1)),$$

so by unicity of the decomposition in irreducible factors and as $i \notin \mathbb{R}$, $F$ is irreducible in $\mathbb{R}[X, Y]$.

However, $V(F) = V(Y) \cap V(X(X-1)) = \{(0,0), (1,0)\}$

**Exercise** 3.2.

(1) Consider the twisted cubic curve $C = \{(t, t^2, t^3); t \in \mathbb{C}\} \subset \mathbb{A}^3(\mathbb{C})$. Show that $C$ is an irreducible closed subset of $\mathbb{A}^3(\mathbb{C})$. Find generators for the ideal $I(C)$.
(2) Let $V = V(X^2 - YZ, XZ - X) \subset \mathbb{A}^3(\mathbb{C})$. Show that $V$ consists of three irreducible components and determine the corresponding prime ideals.

**Solution 2.**

(1) The idea is to write $C$ as the image of some morphism and use the fact that the continuous image of an irreducible topological space is irreducible. The map

$$\begin{aligned} f\colon \mathbb{A}^1(\mathbb{C}) &\to \mathbb{A}^3(\mathbb{C}) \\ t &\mapsto (t, t^2, t^3). \end{aligned}$$

is a morphism of algebraic sets (induced by the $\mathbb{C}$-algebra morphism $\Phi\colon \mathbb{C}[X, Y, Z] \to \mathbb{C}[X]$ mapping $X \mapsto X$, $Y \mapsto X^2$, $Z \mapsto X^3$), so in particular it is continuous. Clearly, the image of $f$ is $C$. If $C = F_1 \cup F_2$, with $F_1$ and $F_2$ Zariski closed subsets, $\mathbb{A}^1 = f^{-1}(F_1) \cup f^{-1}(F_2)$. $\mathbb{A}^1(\mathbb{C})$ is Zariski-irreducible so without lost of generality $f^{-1}(F_1) = \mathbb{A}^1(\mathbb{C})$ but then $C = F_1$. Hence $C$ is irreducible. Moreover, putting $I = (Y - X^2, Z - X^3)$, it is straightforward to see that $C = V(I)$, so $C$ is closed. Finally, as $I$ is prime (because $I = \ker \Phi$ and thus $\mathbb{C}[X, Y, Z]/I \cong \mathbb{C}[X]$), we have $I(C) = I$.
(2) We can decompose

$$\begin{aligned} V(X^2 - YZ, XZ - X) &= V(X^2 - YZ, X(Z-1)) \\ &= V(X^2 - YZ) \cap V(X(Z-1)) \\ &= V(X^2 - YZ) \cap (V(X) \cup V(Z-1)) \\ &= \big(V(X^2 - YZ) \cap V(X)\big) \cup \big(V(X^2 - YZ) \cap V(Z-1)\big) \\ &= V(X^2 - YZ, X) \cup V(X^2 - YZ, Z-1) \\ &= V(YZ, X) \cup V(X^2 - Y, Z-1) \\ &= V(Y, X) \cup V(Z, X) \cup V(X^2 - Y, Z-1). \end{aligned}$$

It is straightforward to see that $(Y, X)$, $(Z, X)$ and $(X^2 - Y, Z - 1)$ are prime (for the last one, consider the $\mathbb{C}$–algebra morphism $\mathbb{C}[X, Y, Z] \to \mathbb{C}[X]$ sending $X \mapsto X$, $Y \mapsto X^2$ and $Z \mapsto 1$), so the three sets in the above decomposition are irreducible. As none of these sets is contained in the union of the other two, we have found the decomposition into irreducible subsets. In fact, $V(Y, X)$ is the $Z$-axis, $V(Z, X)$ the $Y$-axis and $V(X^2 - Y, Z - 1)$ a parabola in the $Z = 1$ plane.

**Exercise** 3.3. For topological spaces $X$ and $Y$, the opens of the product topology on $X \times Y$ are *unions* of products of opens $U \times V$, where $U \subseteq X$ and $V \subseteq Y$. A topological space $X$ is called *Hausdorff* if for any pair of points $x_1 \neq x_2 \in X$, there exist open subsets $U, V \subseteq X$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. A topological space $G$ with an abstract group structure is called a *topological group* if the multiplication and inverse laws are continuous. Let $n \geq 1$.

(1) Is the product topology on $\mathbb{A}^1_k \times \mathbb{A}^1_k$ (each copy of $\mathbb{A}^1_k$ being endowed with the Zariski topology) the same as the Zariski topology on $\mathbb{A}^2_k$?
(2) Is the Zariski topology on $\mathbb{A}^n_k$ Hausdorff?
(3) Is $(\mathbb{A}^n_k, +)$ a topological group for the Zariski topology (assuming $\mathbb{A}^n_k \times \mathbb{A}^n_k \simeq \mathbb{A}^{2n}_k$ is endowed with the Zariski topology)?

**Solution 3.**

(1) No. The non-trivial closed subsets of $\mathbb{A}^1_k \times \mathbb{A}^1_k$ in the product topology are precisely the subsets of the form $(F_1 \times \mathbb{A}^1_k) \cup (\mathbb{A}^1_k \times F_2)$ for finites subsets $F_1, F_2 \subseteq \mathbb{A}^1_k$. But for example, $V(1 - XY)$ is a Zariski closed subset in $\mathbb{A}^2_k$ but it is not closed in the product topology, as it clearly isn't of the above form. $V(X - Y)$ also works.
(2) No. Since any two non-empty open sets are dense in $\mathbb{A}^n_k$, their intersection cannot be empty. Note that this shows that the Zariski topology on $\mathbb{A}^{2n}_k$ doesn't agree with the product topology on $\mathbb{A}^n_k \times \mathbb{A}^n_k$: indeed, the diagonal $\{(p, p) \mid p \in \mathbb{A}^n_k\}$ is closed in the Zariski topology, but if it were closed for the product topology then $\mathbb{A}^n_k$ would be Hausdorff, contradiction.
(3) Yes. Since $(x, y) \mapsto x + y$ and $x \mapsto -x$ are algebraic (i.e. given by polynomials), it is continuous for the Zariski topology. The corresponding $k$–algebra homomorphisms are

$$\begin{aligned} k[x_1, \ldots, x_n] &\to k[y_1, \ldots, y_n, z_1, \ldots, z_n] \\ x_i &\mapsto y_i + z_i, \end{aligned}$$

resp.

$$\begin{aligned} k[x_1, \ldots, x_n] &\to k[x_1, \ldots, x_n] \\ x_i &\mapsto -x_i. \end{aligned}$$

**Exercise** 3.4.

(1) Show that any open subset of an irreducible topological space is irreducible and dense.
(2) Show that the closure of an irreducible subset of a topological space is irreducible.

**Solution 4.** Recall that in general, a topological space $S$ is irreducible if for all $W_1$, $W_2$ closed in $S$ with $S = W_1 \cup W_2$, one has $S = W_1$ or $S = W_2$.

(1) Let $U \subset F$ open in $F$ irreducible. Then $F = (F \setminus U) \cup \overline{U}$ so $F = F \setminus U$ or $F = \overline{U}$, thus $U = \emptyset$ or $U$ is dense in $F$.

   It remains to show that $U$ is irreducible. Let $W_1, W_2$ be closed in $U$ such that $U = W_1 \cup W_2$. Now $U \setminus W_i$ is open in the open subset $U$ of $F$, so $U \setminus W_i$ is open in $F$. In particular, it is dense or empty. On the other hand, note that

$$(U \setminus W_1) \cap (U \setminus W_2) = U \setminus (W_1 \cup W_2) = \emptyset.$$

   As a finite intersection of dense-opens is dense open, one of $U \setminus W_1$ and $U \setminus W_2$ has to be empty, WLOG $U \setminus W_1 = \emptyset$. That is, $U = W_1$, so we are done.
(2) Let $X$ be a topological space and let $S \subseteq X$ be irreducible (with the subspace topology). Suppose that $\overline{S} = F_1 \cup F_2$ for closed subsets $F_1, F_2 \subseteq \overline{S}$. Then

$$S = (S \cap F_1) \cup (S \cap F_2),$$

   and by irreducibility of assume we WLOG have $S \cap F_1 = S$. But this means $S \subseteq F_1$, and as $F_1$ is closed in $\overline{S}$ and thus also in $X$, we obtain $\overline{S} \subseteq F_1$, and thus $\overline{S} = F_1$.

**Exercise** 3.5. Let $V$ an affine variety. Show that algebraic subsets of $V$ are in one-to-one correspondence with radical ideals of $\Gamma(V)$. Show that under this correspondence, affine subvarieties correspond to prime ideals and points correspond to maximal ideals.

**Solution 5.** We have seen in the lecture this correspondence for $V = \mathbb{A}_k^n$, given by

$$\begin{aligned}\{\text{Algebraic subsets of } \mathbb{A}_k^n\} &\longleftrightarrow \{\text{Radical ideals in } k[x_1,\dots,x_n]\}\\ W &\longmapsto I(W)\\ V(J) &\longleftarrow\!\shortmid J,\end{aligned}$$

where irreducibles correspond to primes and points correspond to maximal ideals. By definition, the algebraic subsets of $V$ are the algebraic subsets of $\mathbb{A}_k^n$ contained in $V$. Hence, if $W \subseteq V$ is an algebraic subset of $V$, then $I(W)$ is radical and $I(W) \supseteq I(V)$. On the other hand, if $J$ is radical and $J \supseteq I(V)$, then we have $V(J) \subseteq V$. Therefore, the above correspondence restricts and co-restricts to a one-to-one correspondence

$$\begin{aligned}\{\text{Algebraic subsets of } V\} &\longleftrightarrow \{\text{Radical ideals in } k[x_1,\dots,x_n] \text{ containing } I(V)\}\\ W &\longmapsto I(W)\\ V(J) &\longleftarrow\!\shortmid J,\end{aligned}$$

where again irreducibles correspond to primes and points correspond to maximal ideals. As we also have a one-to-one correspondence

$$\begin{aligned}\{\text{ideals in } k[x_1,\dots,x_n] \text{ containing } I(V)\} &\longleftrightarrow \{\text{ideals in } \Gamma(V) := k[x_1,\dots,x_n]/I(V)\}\\ J &\longmapsto \pi(J)\\ \pi^{-1}(J) &\longleftarrow\!\shortmid J,\end{aligned}$$

where $\pi\colon k[x_1,\dots,x_n] \to \Gamma(V)$ is the projection, and as this correspondence preserves radical, prime and maximal ideals by Rings&Modules, we obtain a one-to-one correspondence

$$\begin{aligned}\{\text{Algebraic subsets of } V\} &\longleftrightarrow \{\text{Radical ideals in } \Gamma(V)\}\\ W &\longmapsto \pi(I(W))\\ V(\pi^{-1}(J)) &\longleftarrow\!\shortmid J,\end{aligned}$$

whereby irreducibles correspond to primes and points correspond to maximal ideals.

**Remark.** Note that the elements of $\Gamma(V)$ can be seen as the algebraic functions $V \to k$, because if $\overline{f} \in \Gamma(V)$ is represented by $f \in k[x_1,\dots,x_n]$, then $f(p) = g(p)$ for all $g$ with $g \in f + I(V)$ and $p \in V$, hence it makes sense to define $\overline{f}(p) := f(p)$ for any $p \in V$. For a subset $S \subseteq \Gamma(V)$, one can then define

$$V_{\Gamma(V)}(S) := \{p \in V \mid \forall \overline{s} \in S:\ \overline{s}(p) = 0\},$$

and for any subset $W \subseteq V$, we can define

$$I_{\Gamma(V)}(W) := \{\overline{f} \in \Gamma(V) \mid \forall p \in W:\ \overline{f}(p) = 0\}.$$

With these definitions, the above correspondence can be written as

$$\begin{aligned} \{\text{Algebraic subsets of } V\} &\longleftrightarrow \{\text{Radical ideals in } \Gamma(V)\} \\ W &\longmapsto I_{\Gamma(V)}(W) \\ V_{\Gamma(V)}(J) &\longleftarrow\!\shortmid J, \end{aligned}$$

in perfect analogy to the case $V = \mathbb{A}^n_k$.